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A METHOD FOR CALCULATING THE DYNAMICS OF A MASSIVE PUNCH ON A LAMINATED BASE†

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A method for solving a non-steady plane contact problem for a laminated medium is proposed. It is based on a combination of analytical methods (the virtual absorption method and the vector-valued eigenfunction method for defining the influence function of the medium in integral Fourier transforms) and on numerical inversion of Laplace transforms. The advantage of the method is that it offers the possibility of examining systems with an arbitrarily number of layers. The solution can be used to study the dynamics of the punch in detail and to trace the changes in the nature of the non-steady process in the medium.

Numerical analysis is carried out for a band-shaped punch in frictionless contact with a stack of three or more layers rigidly attached to a non-deformable base. A vertical load is applied to the punch. The displacement of the punch and of the points of the medium are analysed as functions of time, for different relationships among the elastic and geometric parameters of the problem.

1. STATEMENT OF THE PROBLEM AND SCHEME OF SOLUTION

CONSIDER the dynamical interaction of a rigid band-shaped punch of mass M and width $2a$ with a laminated semi-bounded medium. A load $\mathbf{P}(t)$ which varies arbitrarily with time t, is applied to the punch. The medium occupies the region $-H \leqslant z \leqslant 0$, $-\infty \leqslant x$, $y \leqslant \infty$ and is represented as a stack of N rigidly interconnected layers of overall thickness $H = 2(h_1 + \ldots + h_N)$ (h_i denotes half the thickness, μ_i is the Lamé parameter, ν_i is Poisson's ratio and ρ_i is the density of the ith layer). The lower surface of the stack is rigidly attached to a non-deformable base. The system is initially at rest.

In a plane setting, the displacements of an elastic medium $\mathbf{u}(x, z, t) = \{u_1, u_2\}$ due to an arbitrary load $\mathbf{q}(x, t) = \{q_1, q_2\}$ given in some region $|x| \leqslant a$, $z = 0$, may be expressed as convolutions [1] in Laplace transforms with respect to the time t

$$\mathbf{u}(x,z,p) = \int_{-a}^{a} \mathbf{k}(x-\xi, z, p)\mathbf{q}(\xi, p)d\xi \tag{1.1}$$

$$\mathbf{k}(x,z,p) = (2\pi)^{-1} \int_{-\infty}^{\infty} \mathbf{K}(\alpha, z, p)e^{-i\alpha x}d\alpha \tag{1.2}$$

where p is the Laplace transform parameter and α is the Fourier transform parameter.

Equations (1.1) and (1.2), which are derived by means of integral transforms [1] from Lamé's differential equations for the motion of the points of the medium, may be written differently as

$$\mathbf{U}(\alpha, z, p) = \mathbf{K}(\alpha, z, p)\mathbf{Q}(\alpha, p) \tag{1.3}$$

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where the matrix function $\mathbf{K}(\alpha, z, p) = \| K_{ij} \|_{ij=1}^{2}$ in the integrand is determined by the type of medium; for laminated media it has the same form as in the theory of steady vibrations, except that instead of the vibration frequency ω one puts ip, where i is the square root of -1. The properties of $\mathbf{K}(\alpha, z, p)$ are known [1]; $\mathbf{U}(\alpha, z, p)$, $\mathbf{Q}(\alpha, p)$ are the Fourier and Laplace transforms of the functions u and q, respectively.

Equations (1.1)–(1.3) are of fundamental importance for studying the interaction of surface objects with elastic media and the propagation of waves from surface sources. One of the most important aspects of the problem is the construction of $\mathbf{K}(\alpha, z, p)$ for laminated media.

If mixed boundary conditions are specified on the surface of the medium

$$\mathbf{u}(x,0,p) = \mathbf{w}(p), \quad |x| \leqslant a, \quad z = 0$$
$$\mathbf{q}(x,p) = 0, \quad |x| > a, \quad z = 0$$

Equation (1.1) yields an integral equation for the contact problem in Laplace transforms. The unknown contact stresses in the contact region $|x| \leqslant a$, $z = 0$, are determined from a Fredholm equation of the first kind

$$\mathbf{w}(p) = \int_{-a}^{a} \mathbf{k}(x - \xi, 0, p)\mathbf{q}(\xi, p)d\xi \tag{1.4}$$

If the dynamical load is transmitted to the medium through a rigid punch, one must consider, in addition to (1.4) differential equations for the motion of the punch itself.

We will outline a general scheme to handle this type of problem, on the assumption that the load is applied vertically to the punch, $\mathbf{P}(t) = \{0, P\}$, and that the contact between the punch and the medium is frictionless. We then have one equation of motion for the punch, which may be written in terms of Laplace transforms

$$Mp^2 w = P(p) - Q(p), \quad Q(p) = \int_{a}^{a} q(x, p)dx, \tag{1.5}$$

where $Q(p)$ is the reaction of the base, which may be written in terms of the functional $Q(\alpha, p)$ as $Q(p) \equiv Q(0, p)$.

1. Construct the matrix-function $\mathbf{K}(\alpha, z, \omega)$ for a laminated medium; substitute $\omega = ip$ to get $\mathbf{K}(\alpha, z, p)$. To allow for viscosity in the medium, make the formal substitution $\omega = ipe^{-i\zeta}$, where ζ is a viscosity parameter for the medium (the coefficient of loss due to internal friction in the material of the base; in that case, the constants of elasticity in the Lamé equations are complex quantities of the form $\lambda_i e^{2i\zeta}$, $\mu_i e^{2i\zeta}$, $0 \leqslant 2\zeta \leqslant 1$ [2]).

2. Use the virtual absorption method to determine the unknown contact stresses $q_0(x, p)$, the reaction of the base $Q_0(p)$ and the functional $Q_0(\alpha, p)$ from the integral equation (1.4), on the assumption that the vertical displacement of the punch is unity, $w = \{0, 1\}$, in which case the equation is

$$\int_{-a}^{a} k(x - \xi, p)q_0(\xi, p)d\xi = 1 \tag{1.6}$$

$$k(x, p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} K(\alpha, p)e^{-i\alpha x}d\alpha, \quad K(\alpha, p) = K_{22}(\alpha, 0, p) \tag{1.7}$$

3. Use the equation of motion (1.5) and the linear formulation of the problem to determine the punch displacement $\mathbf{w} = \{0, w\}$

$$w(p) = P(p)(Mp^2 + Q_0(p))^{-1} \tag{1.8}$$

4. The displacements of the points of the medium for $|x| > \alpha$, $-H \leqslant z \leqslant 0$, are obtained in terms of Laplace transforms from (1.3), where

$$Q(\alpha,p)=w(p)Q_0(\alpha,p) \tag{1.9}$$

5. Inverting transforms in (1.8), (1.3) and (1.9), we obtain the required functions $W(t)$ and $\mathbf{u}(x,z,t)$, representing the motion of the rigid punch and the displacements in the medium.

2. CONSTRUCTION OF $\mathbf{K}(\alpha, z, \omega)$ FOR A STACK OF N LAYERS

Consider an elastic layer of thickness $2h$, $-h\leqslant z\leqslant h$, with harmonic loads $\mathbf{T}e^{-i\omega t}$, $\mathbf{R}e^{-i\omega t}$, applied to its upper and lower faces, respectively. The solution for the layer is constructed by the method of vector-valued eigenfunctions [3]; in three dimensions, it is

$$\mathbf{u}(x,y,z)=\frac{e^{-i\omega t}}{16\pi^2}\int\int_{-\infty}^{\infty}\mathbf{a}(z)e^{-i(\alpha x+\beta y)}d\alpha d\beta \tag{2.1}$$

$$\mathbf{u}(x,y,z)=\{u_1,u_2,u_3\}$$

Write the solution in terms of Fourier transforms with parameters α, β ($\mathbf{Q}^{\pm}=\{Q_1^{\pm},\ Q_2^{\pm},\ Q_3^{\pm}\}$ is the load vector)

$$\mathbf{a}(z)=[\mathbf{A}_h^+(z)\mathbf{Q}^+ +\mathbf{A}_h^-(z)\mathbf{Q}^-]\mu^{-1}$$

$$Q_1^{\pm}=i\lambda^{-2}[(t_1\mp r_1)\alpha+(t_2\mp r_2)\beta]$$

$$Q_2^{\pm}=i\lambda^{-2}[(t_1\mp r_1)\beta-(t_2\mp r_2)\alpha],\quad Q_3^{\pm}=t_3\pm r_3$$

$$\mathbf{t}=\int\int_{-\infty}^{\infty}\mathbf{T}e^{i(\alpha x+\beta y)}dxdy,\quad \mathbf{r}=\int\int_{-\infty}^{\infty}\mathbf{R}e^{i(\alpha x+\beta y)}dxdy.$$

where the elements of the matrix $\mathbf{A}_h^{\pm}$ are

$$\begin{aligned}
&A_{11}^+=i\alpha(\Delta^+)^{-1}\sigma_2(\lambda^2\operatorname{ch}(z\sigma_1)\operatorname{ch}(h\sigma_2)-\gamma\operatorname{ch}(h\sigma_1)\operatorname{ch}(z\sigma_2))\\
&A_{12}^+=-i\beta 2\operatorname{ch}(z\sigma_2)(\sigma_2\operatorname{sh}(h\sigma_2))^{-1}\\
&A_{13}^+=-i\alpha(\Delta^+)^{-1}(\gamma\operatorname{ch}(z\sigma_1)\operatorname{sh}(h\sigma_2)-\sigma_1\sigma_2\operatorname{sh}(h\sigma_1)\operatorname{ch}(z\sigma_2))\\
&A_{21}^+=\beta/\alpha A_{11}^+,\quad A_{22}^+=-\alpha/\beta A_{12}^+,\quad A_{23}^+=\beta/\alpha A_{13}^+\\
&A_{31}^+=\lambda^2(\Delta^+)^{-1}(\gamma\operatorname{ch}(h\sigma_1)\operatorname{sh}(z\sigma_2)-\sigma_1\sigma_2\operatorname{sh}(z\sigma_1)\operatorname{ch}(h\sigma_2))\\
&A_{32}^+=0\\
&A_{33}^+=\sigma_1(\Delta^+)^{-1}(\gamma\operatorname{sh}(z\sigma_1)\operatorname{sh}(h\sigma_2)-\lambda^2\operatorname{sh}(h\sigma_1)\operatorname{sh}(z\sigma_2))\\
&\Delta^+=\gamma^2\operatorname{ch}(h\sigma_1)\operatorname{sh}(h\sigma_2)-\lambda^2\sigma_1\sigma_2\operatorname{sh}(h\sigma_1)\operatorname{ch}(h\sigma_2)\\
&\lambda^2=\alpha^2+\beta^2,\quad \gamma=\lambda^2-1/2\theta_2^2,\quad \sigma_i^2=\lambda^2-\theta_i^2\\
&\theta_1^2=\varepsilon\theta_2^2,\quad \theta_2^2=\rho\omega^2/\mu,\quad \varepsilon=(1-2\nu)/(2-2\nu)
\end{aligned} \tag{2.2}$$

The matrix $\mathbf{A}_h^-(z)$ and Δ^- are found by cyclic substitution of sh for ch in $\mathbf{A}_h^+(z)$ and Δ^+; μ is the Lamé parameter, ν is Poisson's ratio, ρ is the density of the layer, and ω is the vibration frequency.

We define two matrices of a special structure

$$\mathbf{C}^{\pm}=\left\|\begin{matrix}\pm i\alpha\lambda^{2} & \pm i\beta\lambda^{2} & 0\\ \pm i\beta\lambda^{2} & \pm i\alpha\lambda^{2} & 0\\ 0 & 0 & 1\end{matrix}\right\|$$

Then $\mathbf{Q}^{\pm}=\mathbf{C}^{+}\mathbf{t}\pm\mathbf{C}^{-}\mathbf{r}$, and the solution for one layer may be written as follows:

$$\mathbf{a}(z)=(\mathbf{A}(z)\mathbf{t}+\mathbf{B}(z)\mathbf{r})\mu^{-1} \tag{2.3}$$

$$\mathbf{A}(z)=[\mathbf{A}_h^{+}(z)+\mathbf{A}_h^{-}(z)]\mathbf{C}^{+},\quad \mathbf{B}(z)=[\mathbf{A}_h^{+}(z)-\mathbf{A}_h^{-}(z)]\mathbf{C}^{-} \tag{2.4}$$

If the medium is a stack of N rigidly interconnected layers of thickness $H=2(h_1+\ldots+h_N)$ (where h_i is half the thickness of the ith layer), with a rigidly attached lower face, the overall solution may be obtained with the help of the single-layer solution (2.1), (2.3).

We first introduce local coordinates for each layer

$$z_k=z+2(h_1+\ldots+h_{k-1})+h_k,\quad k=1,2,\ldots,N$$

Formally separating the layers, we can express the displacements of the points of layer k as follows:

$$\mathbf{a}_k(z_k)=(\mathbf{A}(z_k)\mathbf{t}_{k-1}+\mathbf{B}(z_k)\mathbf{t}_k)\mu_k^{-1},\quad k=1,2,\ldots,N \tag{2.5}$$

where $\mathbf{t}_k$ are forces representing the interaction between layers, and $\mathbf{t}_0$ is the force acting on the upper face.

Note that in calculating the elements of the matrices $\mathbf{A}(z_k)$, $\mathbf{B}(z_k)$ in (2.5), as defined in (2.2) and (2.4), we must take $h=h_k$, $\mu=\mu_k$, $\rho=\rho_k$, $\nu=\nu_k$.

The displacement must satisfy the joining conditions

$$\mathbf{a}_k(-h_k)=\mathbf{a}_{k+1}(h_{k+1}),\quad k=1,2,\ldots,N-1 \tag{2.6}$$

and the rigid attachment condition

$$\mathbf{a}_N(-h_N)=0 \tag{2.7}$$

Formulae (2.6) imply the following recurrence relation

$$\mathbf{A}(-h_k)\mathbf{t}_{k-1}+[\mathbf{B}(-h_k)-g_k\mathbf{A}(h_{k+1})]\mathbf{t}_k=g_k\mathbf{B}(h_{k+1})\mathbf{t}_{k+1} \tag{2.8}$$

$$g_k=\mu_k/\mu_{k+1}$$

From (2.7) we determine

$$\mathbf{t}_N=-\mathbf{B}^{-1}(-h_N)\mathbf{A}(-h_N)\mathbf{t}_{N-1} \tag{2.9}$$

Using the recurrence relations (2.8) and (2.9), we can express the forces $\mathbf{t}_k$ in terms of the surface load $\mathbf{t}_0$

$$\mathbf{t}_k=(-1)^k\Pi_{i=k}^{1}\mathbf{F}_i^{-1}\mathbf{A}(-h_i)\mathbf{t}_0,\quad k=1,2,\ldots,N$$

$$\mathbf{F}_N=\mathbf{B}(-h_N),\quad \mathbf{F}_k=\mathbf{B}(-h_k)-g_k\mathbf{A}(h_{k+1})+g_k\mathbf{B}(h_{k+1})\mathbf{F}_{k+1}^{-1}\mathbf{A}(-h_{k+1}),$$

$$k=1,2,\ldots,N-1$$

Thus, the displacements of the points of the laminated medium may be written as

$$\begin{aligned}
&\mathbf{a}(z) = \mathbf{K}(\alpha,\beta,z,\omega)\mathbf{t}_0 \\
&\mathbf{K}(\alpha,\beta,z,\omega) = (-1)^{k-1}(\mathbf{A}(z_k) - \mathbf{B}(z_k)\mathbf{F}_k^{-1}\mathbf{A}(-h_k))*\Pi_{i=k-1}^{1}\mathbf{F}_i^{-1}\mathbf{A}(-h_i)\mu_k^{-1}, \\
&k = 1,2,\ldots,N
\end{aligned} \tag{2.10}$$

In particular, it follows from (2.10) that for one layer rigidly attached to a non-deformable base

$$\mathbf{a}(z) = (\mathbf{A}(z+h_1) - \mathbf{B}(z+h_1)\mathbf{B}^{-1}(-h_1)\mathbf{A}(-h_1))\mathbf{t}_0\mu_1^{-1}$$

For a stack of two layers the displacements are, in the upper layer

$$\mathbf{a}(z) = (\mathbf{A}(z+h_1) - \mathbf{B}(z+h_1)\mathbf{F}_1^{-1}\mathbf{A}(-h_1))\mathbf{t}_0\mu_1^{-1}$$

in the lower layer

$$\begin{aligned}
&\mathbf{a}(z) = -(\mathbf{A}(z_2) - \mathbf{B}(z_2)\mathbf{B}^{-1}(-h_2)\mathbf{A}(-h_2))\mathbf{F}_1^{-1}\mathbf{A}(-h_1)\mathbf{t}_0\mu_2^{-1} \\
&z_2 = z + 2h_1 + h_2
\end{aligned}$$

where

$$\mathbf{F}_1 = \mathbf{B}(-h_1) - g_1\mathbf{A}(h_2) + g_1\mathbf{B}(h_2)\mathbf{B}^{-1}(-h_2)\mathbf{A}(-h_2)$$

It has been shown that as $\lambda \to \infty$ the asymptotic behaviour of the matrix $\mathbf{K}$ at the surface of the medium $(z=0)$ is identical with that of $\mathbf{K}$ for a layer, being given by

$$\mathbf{K}(\alpha,\beta,0,\omega) \approx \mathbf{A}(h_1) = \begin{Vmatrix} \alpha^2 M + \beta^2 S & \alpha\beta(M-S) & -i\alpha L \\ \alpha\beta(M-S) & \beta^2 M + \alpha^2 S & -i\beta L \\ i\alpha L & i\beta L & R \end{Vmatrix}$$

the functions M, S, L and R depend on the frequency ω and the parameter $\lambda^2 = \alpha^2 + \beta^2$; they admit of the following asymptotic representations as $\lambda \to \infty$

$$\begin{aligned}
&M \approx 4(1-\nu_1)|\lambda|^{-3}, \qquad S \approx 4|\lambda|^{-3} \\
&L \approx (4\nu_1 - 2)\lambda^{-2}, \qquad R \approx 4(1-\nu_1)|\lambda|^{-1}
\end{aligned}$$

In two dimensions, we have $\beta = 0$ $(\lambda \equiv \alpha)$; the kernel in (1.2) is obtained by deleting the second column and row from the matrix $\mathbf{K}(\alpha, 0, z, \omega) = \| K_{ij} \|_{ij=1}^{3}$ and substituting $\omega = ip$. Then

$$\begin{aligned}
&\mathbf{K}(\alpha,0,p) \approx \begin{Vmatrix} c|\alpha|^{-1} & -ib\alpha^{-1} \\ ib\alpha^{-1} & c|\alpha|^{-1} \end{Vmatrix}, \qquad \alpha \to \infty \\
&c = 4(1-\nu_1), \qquad b = 4\nu_1 - 2
\end{aligned}$$

The approach outlined above for determining the matrix-symbol $\mathbf{K}(\alpha, z, p)$, unlike other methods [4], enables one to avoid the difficulties encountered in the numerical treatment, due to the presence of increasing exponential constituents in the fundamental solutions of the relevant systems of differential equations. Because of such exponential terms, the linear algebraic systems of equations obtained in dealing with the boundary conditions may well be ill-conditioned.

3. SOLUTION OF THE CONTACT PROBLEM

Let $\mathbf{T}_0\{0,\ q_0(x,\ p)\}$ be a solution of the integral equation (1.6), (1.7) when the right-hand side is unity ($Kq_0=1$). Then the contact stresses in the punch/medium interaction region are given by $q(x,\ p)=w(p)q_0(x,\ p)$. The solution $q_0(x,\ p)$ was constructed by the virtual absorption method of [5, 6], which enables the singularity of the contact pressures on the punch surface to be determined analytically. The Fourier integral $Q_0(\alpha,\ p)$ of $q_0(x,\ p)$ was computed in quadratures. The form of the functions $q_0(x,\ p)$ for fixed values of ω ($\omega=ip$) is known [6].

The function $Q_0(\alpha,\ p)$ obtained by virtual absorption is fairly simple in form (B denotes an approximation parameter, $B \gg 1$ [6]

$$
\begin{aligned}
&Q_0(\alpha,p)=K^{-1}(\alpha,p)(\alpha^2+B^2)^{-1/2}\times\\
&\times\left\{f_2(\alpha)+f_2(-\alpha)-\tfrac{1}{2}\sum_{k=1}^{n}C_k(p)[f_1(\alpha,x_k)+f_1(\alpha,-x_k)]\right\}
\end{aligned}
\tag{3.1}
$$

The coefficients $C_k(p)$ are determined by solving a linear algebraic system of equations ($k,\ l=1,2,\ldots,n$)

$$\sum_{k=1}^{n}C_k(p)[f_1(\psi_l,x_k)+f_1(\psi_l,-x_k)]=2[f_2(\psi_l)+f_2(-\psi_l)]$$

$$f_1(\alpha,x)=(B+i\alpha)^{1/2}e^{i\alpha a}F(\alpha,x)+(B-i\alpha)^{1/2}e^{-i\alpha a}F(-\alpha,-x)$$

$$f_2(\alpha)=(i\alpha)^{-1}e^{i\alpha a}((B+i\alpha)B)^{1/2}\operatorname{erf}[2a(B+i\alpha)]^{1/2}$$

$$F(\alpha,x)=\sum_{j=1}^{n}\operatorname*{Res}_{\alpha=\varphi_j}H(\alpha,p)e^{i\varphi_j}(a-x)(B-i\varphi_j)^{-1/2}\ (\varphi_j+\alpha)^{-1}$$

$$H(\alpha,p)=c^{-1}K(\alpha,p)(\alpha^2+B^2)^{1/2}$$

where x_k are points dividing the interval $(0,\ a)$ into n equal parts, and φ_j and ψ_k are the poles and zeros of $H(\alpha,p)$ in the upper half-plane of α.

4. DISPLACEMENTS OF THE PUNCH AS A FUNCTION OF TIME

After substituting (1.9), (3.1) for $\alpha=0$ into Eq. (1.5), we obtain (1.8). To obtain the final solution, we have to invert the Laplace transforms

$$W(t)=\frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty}w(p)e^{pt}dp,\qquad \xi>0$$

Physical considerations dictate that the integrand should not have roots in the upper half-plane $\operatorname{Re}p>0$, so the integral along a straight line parallel to the imaginary axis may be replaced by the integral along the imaginary axis. The substitution $p=-i\omega$ then converts the Laplace inversion integral into a Fourier integral

$$W(t)=\frac{2}{\pi}\int_0^\infty \operatorname{Re}w(i\omega)\cos\omega t\,d\omega=-\frac{2}{\pi}\int_0^\infty \operatorname{Im}w(i\omega)\sin\omega t\,d\omega \tag{4.1}$$

To compute the base reaction $Q(t)$ and the normal stresses $q(x, t)$ in the contact region, we have to replace the function $w(p)$ in the integrand of (4.1) by $w(p)Q_0(p)$ and $w(p)q_0(x, p)$, respectively.

In what follows, we use Filon's quadrature formula to compute integrals of the type of (4.1), as it yields rapid and accurate results for integrals of oscillating functions.

5. DISPLACEMENTS OF POINTS OF THE MEDIUM

The displacements of the points of the medium in the region $|x|>a$, $-H\leqslant z\leqslant 0$, are obtained in Fourier–Laplace transforms from formula (1.3), with due attention to (1.9). Inverting the Fourier transforms we get

$$u_j(x,z,p)=\frac{w(p)}{\pi}\int_0^\infty K_{j2}(\alpha,z,p)Q_0(\alpha,p)\cos(\alpha x)d\alpha,\quad j=1,2$$

The required displacements $u_j(x, z, t)$ of the points of the medium are now obtained from (4.1) with $w(p)$ replaced by $u_j(x, z, p)$.

6. NUMERICAL ANALYSIS

A numerical analysis of the problem was undertaken for a band-shaped punch in frictionless contact with a stack of three or more layers rigidly attached to a non-deformable base. The behaviour of the system was investigated for varying geometric and elastic parameters of the layers: the thickness, stiffness, density and viscosity of the medium. A comparative study of the behaviour of the punch was carried out for three, five and seven layers.

Figure 1 plots the displacements of a punch of unit mass in frictionless contact with a three-layer medium for the following parameter values: $\mu_1=1$, $\mu_2=0.25$ and $\mu_3=0.5$. A pulsed load $P(t)=H(t)-H(t-0.01)$ is applied to the punch. Curves 1–3 are drawn for different thicknesses of the upper layer: $2h_1=0.4$, 0.8 and 1.2, but fixed thickness of the second and third layers: $2h_2=0.6$ and $2h_3=0.4$. As the thickness of the upper layer, which is in contact with the softer second layer, increases, one observes an increase in the arrival time of a wave reflected from the interface of the first and second layer. This wave arrives in phase, causing an increase in the punch displacements. The increase in the overall thickness of the stack also increases the period of the natural vibrations of the system after removal of the load.

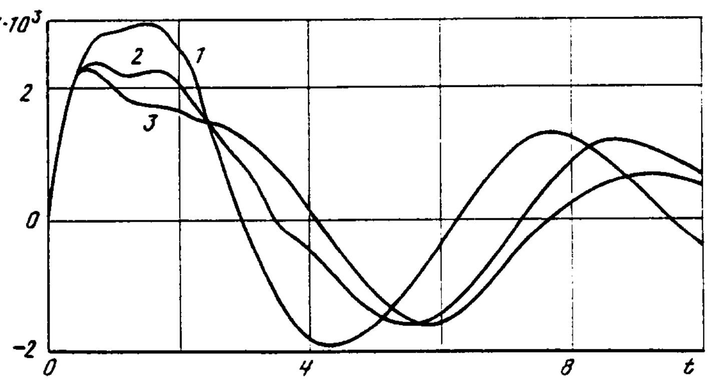

FIG. 1.

Figure 2 plots the displacements of the punch for varying thickness and density of the third (lowest) layer, with the elastic and geometric parameters of the first and second layers fixed $(\mu_1;\ 2h_1) = (1;\ 0.4)$, $(\mu_2;\ 2h_2) = (2;\ 0.6)$. The solid curves correspond to thicknesses $2h_3 = 0.4$, 0.8 and 1.2 (curves 1–3) for $\mu_3 = 0.5$, and the dashed curves to $\mu_3 = 0.5$, 1 and 1.5 (curves 2, 4, 5) for $2h_3 = 0.8$. It can be seen that the punch displacements observed for varying thickness of the lower layer are the same until the return of the wave reflected from the rigid base, in the case of the thinnest layer. As the thickness of the lower layer—which is softer than the second $(\mu_3 < \mu_1 < \mu_2)$—increases, the return of the wave in phase from the interface of the second and third layers becomes more pronounced. The wave reflected from the interface of the first and second layers causes an increase in the punch displacements, because $\mu_1 / \mu_2 = 0.5$, whereas in Fig. 1 $\mu_1 / \mu_2 = 4$. If the stiffness of the lower layer is increased (the dashed curves), keeping its thickness fixed, a decrease occurs in the period of the system's natural vibrations and the punch reaches equilibrium at an earlier time.

Figure 3 plots $W(t)$ for stacks of three, five and seven layers with each layer of fixed thickness $2h_i = 0.2$ and stepwise varying stiffness. The solid curves are plotted for $\mu_i = 1 + (i-1)0.1$ and the dashed curves for $\mu_i = 1 - (i-1)0.1$. It is noteworthy that the decrease in stiffness with depth produces an increase in the period of the system's vibrations, and vice versa. Addition of layers also increases the period.

Figure 4 shows the displacements of the punch $z = 0$ and the points of the medium beneath it at depths $z = -0.6$, -1.2 and -1.9. The system consists of two layers of equal thickness $2h_1 = 2h_2 = 1$ and stiffness ratio $\mu_1 / \mu_2 = 4$. Clearly, this model yields a good picture of the physical nature of wave propagation in the medium, both while the load is acting and after its removal. It should be noted that as the depth increases the amplitude of the vibrations decreases and all points of the medium vibrate in phase. As the depth increases, the delay time (in the arrival of the wave) increases, and during that time the medium is at rest. The dashed curves correspond to a thickness $H = 2h_1 = 2$ and $\mu = \mu_1 = 1$, at depths $z = 0$, -0.6, -1.2 and -1.9.

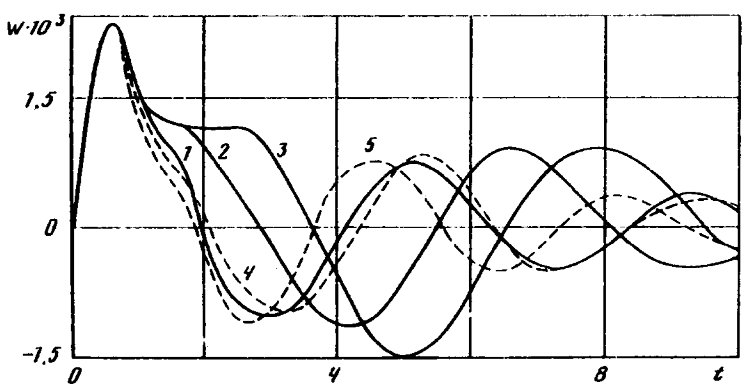

FIG. 2.

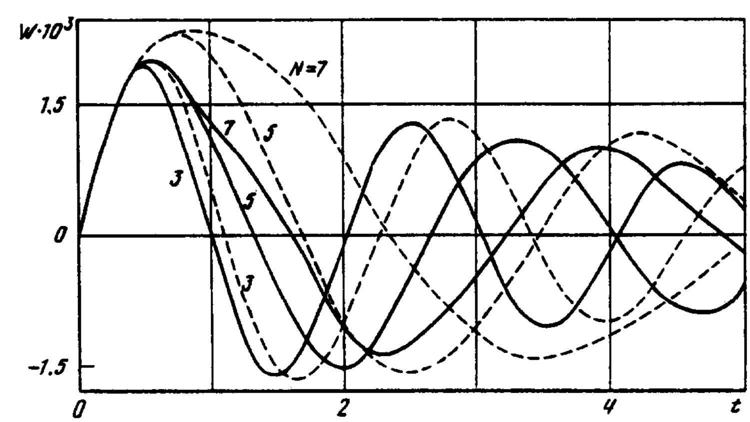

FIG. 3.

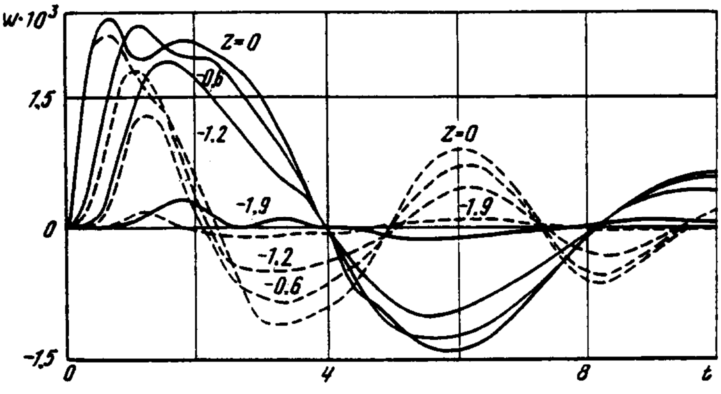

FIG. 4.

All parameters represented in the figures are dimensionless. The displacements are computed in units of half the thickness of the punch α, the load in units of the stiffness of the upper layer μ_1, and the time in units of $(\rho_1/\mu_1)^{1/2}\alpha$.

In all examples the other dimensionless parameters have values $\rho_i = 1$, $\nu_i = 0.3$. The load is $P(t) = H(t) - H(t-0.01)$, the coefficient of friction in the medium $\zeta = 0.2$ and the punch mass $M = 1$.

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